is also normal to the incident flow at orbital perigee. Consequently, on these orbits the satellite encounters maximum aerodynamic resistance. As a result of such a movement of the satellite in the perigee and in the apogee, its bottom part must face the earth periodically, this period being equal to the period of "tumbling."

A slow variation of the parameters of rotation and orientation of the satellite can be explained by the influence of perturbing factors: the effect of aerodynamic and gravitational perturbations (2,4), the interaction of the electrical circuits in the satellite with the earth's magnetic field, and, to a lesser degree, the effect of Foucault (5) currents, and other perturbing factors.

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Reviewer's Comment

The substance of this paper appears to be primarily the solution of a coordinate conversion problem done in some detail and probably quite well, but overlong by usual standards for publication in an American journal.

Information about the instrumentation by which the earth's magnetic field is instantaneously tracked, as well as the angular measuring techniques between different axis systems, would be of interest, but this is not included in the paper. It is possible that some sleuthing would lead to characteristics of Sputnik III as revealed by the minor excursions in its orbit which appear to be disclosed in good faith.

None of us at MIT wishes to vouch for the detailed accuracy of the analysis, but the paper seems to be a high quality effort within the limited area of interest noted.

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Forms of Solutions of Einstein's Equations

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Obviously, there is altogether too great a variety of methods and additional assumptions, unconnected with the boundary conditions, now being used to solve Einstein's equations. This paper is an attempt to determine whether all these forms of solution are equivalent. It is found that some solutions contradict the basic postulates of tensor analysis when considered from the point of view of the two-metric theory of the gravitational field.[1,2]

IN the general theory of relativity the gravitational field is regarded as an effect of the "deviation" of the real from the planar (Euclidian) space-time manifold. The latter is viewed as the physical abstraction of a space devoid of material sources. This fundamental idea is formulated with sufficient completeness in the so-called two-metric theory, which makes explicit use of the characteristics of two spaces, an auxiliary (planar) and a real (Riemannian) space. If $\mathring{g}_{\mu\nu}$ is the metric tensor of a space without sources and $g_{\mu\nu}$ that of a space with sources, then the formation of a gravitational field must be viewed as the process of transformation (mapping) of $\mathring{g}_{\mu\nu}$ into $g_{\mu\nu}$. Both tensors are essentially functions of the same coordinates; hence the properties and laws of transformation of the coordinates are not affected by the mapping.

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In Einstein's ordinary theory, the characteristics of the starting ("empty") space play no part. In this connection, serious difficulties arise in interpreting the physical significance of certain geometrical objects, and this creates a fertile soil for speculations about the choice of a system of coordinates.[3] If, however, we analyze the positive results of this theory, from the point of view of the two-metric formalism, in every case we can detect a tendency to realize the forementioned basic idea.

Let us now consider a few of the methods used in solving gravitational equations.

In the so-called weak field approximation[4] the metric tensor of the unknown space $g_{\mu\nu}$ is taken in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric (for a corresponding choice of coordinates), and $h_{\mu\nu}$ are small "corrections," approximately determined from the equations of gravitation. Here it is a question not only of comparing the unknown and starting (known) spaces but also of predetermining the properties of the starting space.

In the case of spherically symmetric problems, the most general expression of a differential quadratic form can be written

$$ds^2 = f_1 dr^2 + f_2 r^2 d\theta^2 + f_3 r^2 \sin^2\theta d\varphi^2 + f_4 dt^2 \quad (2)$$

(where r, θ, φ, and t are ordinary spherical coordinates and time). Thus each of the components of the unknown tensor is sought in the form of the product of the corresponding known component and a certain function, for example: $g_{22} = f_2 r^2 = f_2 \mathring{g}_{22}$.

In the more general case the transformation employed to obtain Eq. (2) may be represented as

$$g_{\mu\nu} = f_{(\mu\nu)} \mathring{g}_{\mu\nu} \quad (3)$$

where summation is assumed not to apply to the subscripts in parentheses. The quantities $f_{(\mu\nu)}$ do not and cannot constitute a tensor [otherwise Eq. (3) would be degenerate]; accordingly they must be regarded as a set of independent scalar functions. Moreover, it follows from (3) that by choosing a suitable system of coordinates, the metric tensor of planar space $\mathring{g}_{\mu\nu}$, and with it $g_{\mu\nu}$, can be reduced to the diagonal form, and this, in its turn, makes it possible to draw a clear boundary between the "space" $f_{(ii)}$ and "time" $f_{(44)}$ mapping functions.

The best known solutions of gravitational equations are based on the idea of diagonalization of the metric tensor. Moreover, they make use of different assumptions in relation to the functions $f_{(\mu\nu)}$. For example, in Schwarzschild's solution it is assumed that $f_{22} = f_{33} = 1$. If we put $f_{11} = f_{22} = f_{33}$ and denote by dl a space element in planar space, we get the metric

$$ds^2 = f_1 dl^2 + f_2 dt^2 \quad (4)$$

corresponding to so-called "isotropic coordinates." Restricting the functions f_1 and f_2 further, by putting $f_1 = 1/f_3 = f$, we get the quadratic form

$$ds^2 = -f dl^2 + (1/f) dt^2 \quad (5)$$

used by Papapetrou[5] and other authors as the basis of a new (essentially scalar) theory of the gravitational field. Finally, there is the method of conformal representation, in which all the functions $f_{(\mu\nu)}$ are assumed to be equal, so that elements in Riemannian and planar space are connected by the simple relation

$$ds^2 = f d\mathring{s}^2 \quad (6)$$

Since we actually make use of the formalism of the two-metric theory, it is easy to account for the principle of transformation in the forementioned examples. The following important circumstance also leads to the same conclusion: in expressions (2–5) the mapping functions $f_{(\mu\nu)}$ and the components of the metric tensor of planar space $\mathring{g}_{\mu\nu}$ are functions of the same coordinates. Accordingly, the quantities $g_{\mu\nu}$ will be functions of these same coordinates, as confirmed by the method of mapping, irrespective of debatable problems concerning the properties of the "empty" space, the physical significance of Riemann coordinates, etc.

The fact that both tensors $g_{\mu\nu}$ and $\mathring{g}_{\mu\nu}$ belong to the class of functions with the same coordinates makes it possible to solve the problem of the admissible forms of solutions of Einstein's equations. In fact, the determination of the metric of the Riemannian space $g_{\mu\nu}$ reduces essentially to the construction from the given tensor $\mathring{g}_{\mu\nu}$ of a new tensor of the same (second) rank. It is known that for sufficiently regular mapping functions this can be realized in accordance with the scheme[6]

$$g_{\mu\nu} = \varphi \mathring{g}_{\mu\nu} + h_{\mu\nu} \quad (7)$$

(where φ is a scalar, $h_{\mu\nu}$ a tensor of rank 2), with different particular values of the quantities φ and $h_{\mu\nu}$.

We must emphasize that the new second-rank tensor cannot be obtained by multiplying the old one by the independent scalar functions $f_{(\mu\nu)}$ in accordance with (3). In fact, if the set of functions $g_{\mu\nu}$ from (3) form a tensor, then in the new system of coordinates x_α' the equality

$$g_{\mu\nu}' = f_{(\mu\nu)}' \mathring{g}_{\mu\nu}' \quad (8)$$

ought to be observed. Using the law of transformation of $\mathring{g}_{\mu\nu}$ and bearing in mind that $f_{(\mu\nu)}' = f_{(\mu\nu)}$ as a scalar, we get

$$g_{\mu\nu}' = f_{(\mu\nu)} \mathring{g}_{\alpha\beta} \frac{\partial x_\alpha}{\partial x_\mu'} \frac{\partial x_\beta}{\partial x_\nu'}$$

This will be a tensor relation only if

$$f_{(\mu\nu)} \mathring{g}_{\alpha\beta} = g_{\alpha\beta}$$

that is, when $f_{(\mu\nu)} = f_{(\alpha\beta)}$. In other words, all the scalar multipliers muut be the same, and this reduces to a conformal mapping.

Conclusions

The foregoing analysis of existing solutions of gravitational equations shows that, from the point of view of two-metric formalism, only two forms of solutions may be regarded as correct: the first of these is expressed by relation (1), the second by the conformal representation (6). Apparently, the general expression (7) is a combination of the two. The first of these forms might be called the method of the gravitational field proper, since the tensor $h_{\mu\nu}$ describes the potentials of such a field.[2] Under these conditions, the usual assumption that $h_{\mu\nu}$ is small need not hold.

The criterion of correctness is taken as the requirement that the $g_{\mu\nu}$, obtained by transformation of the metric tensor of planar space, have tensor properties. Otherwise a fundamental principle of the construction of metric spaces would be violated.

As a rule these properties are not checked, since in constructing an abstract space the set of the required arbitrary functions is taken as the definition of the metric tensor.[7] Consequently, limitations on $g_{\mu\nu}$ appear whenever the unknown Riemannian space is compared with the starting planar one (or another known Riemannian space), i.e., when a transformation method is employed.

It is evident from Eq. (7) that this method does not restrict the generality of $g_{\mu\nu}$, provided that additional assumptions relating to $h_{\mu\nu}$ and φ are not introduced. On the other hand, the field of a metric tensor can be defined without any connection with other metrics, so that the method cannot be considered the only one possible. It must be borne in mind, however, that all geometrical objects in such a space acquire a definite (physical) significance only when compared with corresponding objects in the related planar space.[7] Thus, if geometry is used as the basis of physical theory, it is impossible to depart completely from the two-metric theory of the gravitational field. Under these conditions, our criterion for limiting the variety of solutions of Einstein's equations acquires practical significance.

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